On the automorphisms of numerical power semigroups

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[1] Based on joint work with Salvatore Tringali.

Outline

Power semigroups and power monoids

Unless otherwise stated, all semigroups (sgrps) are written multiplicatively.

The large power sgrp of a sgrp S is the sgrp $\mathcal{P}(S)$ obtained by endowing the *non-empty* subsets of S with the (provably associative) operation

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\, y \in Y\}.$$

The family $\mathcal{P}_{\text{fin}}(S)$ of all non-empty finite subsets of S is a subsemigroup of $\mathcal{P}(S)$. We call it the finitary power semigroup of S.

If M is a monoid with identity 1_M, then $\mathcal{P}(M)$ is itself a monoid with identity $\{1_M\}$ and it is therefore called the large power monoid of M.

Each of the following is a submonoid of $\mathcal{P}(M)$:

- $\mathcal{P}_1(M) := \{X \in P(M) : 1_M \in X\}$, the reduced large power monoid of M.
- $\mathcal{P}_{\text{fin}}(M) := \{X \in \mathcal{P}(M) \colon |X| < \infty\}$, the finitary power monoid of M.
- $\mathcal{P}_{\text{fin},1}(M) := \{X \in P_{\text{fin}}(M) : 1_M \in X\}$, the reduced finitary power monoid of M.

Depending on the context, these structures will be generically referred to as power semigroups or power monoids (shortly, PMs).

We will use $\mathcal{P}_{\text{fin},0}(M)$ instead of $\mathcal{P}_{\text{fin},1}(M)$ when M is written additively.

Numerical semigroups

A numerical semigroup S is a cofinite subsemigroup of $\mathbb{N}$ (the additive monoid of non-negative integers).

- If $0 \in S$, we call S a numerical monoid.

The Frobenius number of a numerical semigroup S is the maximum element of the complement of S in $\mathbb{Z}$ (the additive group of integers).

We illustrate the definition with two canonical examples:

- For every $n \in \mathbb{N}$, the set $\{x \in \mathbb{N} : x \geq n\} \subseteq \mathbb{N}$ is a numerical semigroup, whose Frobenius number is $n - 1$.
- For any two coprime integers $a, b \in \mathbb{N}$, the set $\{ma + nb : m, n \in \mathbb{N}\} \subseteq \mathbb{N}$ is a numerical monoid, whose Frobenius number is $ab - a - b$.

As fundamental objects in additive combinatorics and commutative algebra, numerical semigroups are closely linked to the Frobenius problem, coordinate rings and semigroup algebras.

Origins

Power sgrps made their first *explicit* appearance in a 1950 paper by Ballieu. They are an instance of the more abstract notion of power algebra and were studied quite intensively in the 1980s and 1990s.

A turning point in their history was marked by a 1967 paper by Takayuki Tamura & John Shafer[(2)], who were especially interested in the following:

Problem 1 (or "the Tamura–Shafer problem").

Given a class $\mathcal{O}$ of sgrps, prove/disprove that $\mathcal{P}(H) \cong \mathcal{P}(K)$, for arbitrary $H, K \in \mathcal{O}$, iff $H \cong K$. (Here and later, $\cong$ means "is sgrp-isomorphic to".)

The crux of the problem lies in the "only if" direction, since every (sgrp) isomorphism $f\colon H \to K$ lifts to a global isomorphism $H \to K$ (i.e., to an isomorphism $\mathcal{P}(H) \to \mathcal{P}(K)$) via the map

$$X \mapsto f[X] := \{f(x)\colon x \in X\}.$$

This map is called the augmentation of f.

[(2)] Tamura & Shafer, Math. Japon. **12** (1967), 25–32.

Conjectures and Complementary Problems

More recently, Bienvenu and Geroldinger[(3)] considered the following:

Bienvenu–Geroldinger conjecture

The reduced finitary PM of a numerical monoid S_1 is isomorphic to the reduced finitary PM of a numerical monoid S_2 iff $S_1 = S_2$.

Remark: Two numerical monoids are equal iff they are isomorphic.

The conjecture has eventually prompted Tringali and Yan to propose the next problem, which serves as a natural companion to Problem 1:

Problem 2.

Let $\mathcal{O}$ be a class of monoids. Given $H, K \in \mathcal{O}$, is it true that $\mathcal{P}_{\mathrm{fin},1}(H)$ is isomorphic to $\mathcal{P}_{\mathrm{fin},1}(K)$ if and only if H is isomorphic to K?

(3) P.-Y. Bienvenu and A. Geroldinger, *On algebraic properties of power monoids of numerical monoids*, Israel J. Math. 265 (2025), 867–900.

Chasing the automorphisms

Isomorphism problems in the spirit of Problems 1 and 2 naturally motivate the study of the automorphisms of power sgrps. In particular, this leads to:

Problem 3.

Given a monoid M, "determine" the (sgrp) automorphisms of $\mathcal{P}_{\mathrm{fin},1}(M)$.

For each $f \in \mathrm{Aut}(M)$, the fnc

$$\mathcal{P}_{\mathrm{fin},1}(M) \to \mathcal{P}_{\mathrm{fin},1}(M) \colon X \mapsto f[X]$$

is a well-defined automorphism of $\mathcal{P}_{\mathrm{fin},1}(M)$, called the reduced finitary augmentation of f. An automorphism of $\mathcal{P}_{\mathrm{fin},1}(M)$ is inner if it is the reduced finitary augmentation of an automorphism of M. So, we have a map

$$\Phi \colon \mathrm{Aut}(M) \to \mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(M))$$

sending an automorphism of M to its reduced finitary augmentation. In fact, Φ is an *injective* (*group*) *homomorphism* from $\mathrm{Aut}(M)$ to $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(M))$.

One may ask whether Φ is also *surjective* and hence an isomorphism.

Progress

Trivially, the answer is already no when $M = (\mathbb{N}, +)$, as the reversion map $\mathrm{rev}\colon X \mapsto \max(X) - X$ is an automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$. More interestingly:

Theorem 3.2 in [Tringali & Yan, JCTA 2025]

The reversion map is the only non-trivial automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$.

For a finite abelian group G, Rago proved $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(G)) \cong \mathrm{Aut}(G)$, with the Klein four-group being the only exception, in which case $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},1}(G)) \cong S_3 \times S_3$ (here S_3 is the symmetric group of degree 3).

Together with Tringali, I further investigated:

Problem 4.

What about the analogue of Problem 3 for $\mathcal{P}_{\mathrm{fin}}(\cdot)$?

We show that $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})) \cong \mathbb{Z}_2 \times \mathrm{Dih}_\infty$, where $\mathbb{Z}_2$ denotes the cyclic group of order two and Dih_∞ the infinite dihedral group.

For a numerical semigroup S, Wong et al. showed that $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin}}(S))$ is trivial unless $S = [\![k, \infty]\!]$ for some $k \in \mathbb{N}$, in which case it is cyclic of order 2.

Extending to infinity

While Problems 3 and 4 focus on finite sets, it is natural to ask:

Problem 5.

What about the automorphism group of $\mathcal{P}(S)$ when S is an infinite sgrp, or the automorphism group of $\mathcal{P}_1(H)$ when H is an infinite monoid?

This infinitary extension is unaddressed in the existing literature. This setting gives rise to many new challenges, as several arguments (e.g., of a combinatorial nature) that work in the finitary framework break down in a critical way.

We will focus our attention on:

- the reduced large power monoid $\mathcal{P}_0(H)$ for a numerical monoid H;
- the large power semigroup $\mathcal{P}(S)$ for a numerical semigroup S.

Our main goal is to prove the triviality of their automorphism groups.

Preliminary theorems

Combining several lemmas, we obtain the following preliminary results that may be of independent interest.

Theorem 2.3 in [Tringali & W. 202?]

If H and K are cancellative monoids, then every isomorphism $f : \mathcal{P}_1(H) \to \mathcal{P}_1(K)$ restricts to an isomorphism $\mathcal{P}_{\mathrm{fin},1}(H) \to \mathcal{P}_{\mathrm{fin},1}(K)$.

Theorem 2.5 in [Tringali & W. 202?]

Let H and K be cancellative commutative semigroups. Every isomorphism $f : \mathcal{P}(H) \to \mathcal{P}(K)$ restricts to an isomorphism $\mathcal{P}_{\mathrm{fin}}(H) \to \mathcal{P}_{\mathrm{fin}}(K)$.

Key ideas:

- counting the solutions to the equation $XA = B$.
- the equations $XA = B$ and $f(X)f(A) = f(B)$ have the same number of solutions.

Lemmas for Theorem 2.3

Here, we only sketch the proof of Theorem 2.3. We begin with two lemmas for subsequent arguments.

Lemma 2.1 in [Tringali & W. 202?]

Let A and B be non-empty subsets of a semigroup S. If A contains a right cancellative element $a \in S$ and B is finite, then the equation $XA = B$ has finitely many solutions X over $\mathcal{P}(S)$.

Proof idea: (i) if $XA = B$ for some $X \in \mathcal{P}(S)$, then $Xa \subseteq B$; (ii) the map $X \mapsto Xa$ on $\mathcal{P}(S)$ is injective.

Lemma 3.1 in [Tringali & Yan · Bull. LMS, to appear]

Let M be a monoid, A be a subset of M containing the identity 1_M, and n be an integer greater than 2. Then the equation $XA = A^n$ has at least $2^{|A|-1}$ solutions X over $\mathcal{P}_1(M)$.

Proof idea: Let B be a subset of $A \setminus \{1_M\}$, and define $Q := A^{n-1} \setminus B$. One can verify that $QA = A^n$, and each choice of B yields a distinct solution $X = Q$. The conclusion follows.

Proof sketch of Theorem 2.3

Let f be an isomorphism from $\mathcal{P}_1(H)$ to $\mathcal{P}_1(K)$ and $A \in \mathcal{P}_{\text{fin},1}(H)$.

- By Lemma 2.1, the equation $XA = A^3$ has finitely many solutions X over $\mathcal{P}_1(H)$.
- It follows that the equation $Yf(A) = f(A)^3$ has finitely many solutions Y over $\mathcal{P}_1(K)$.
- On the other hand, Lemma 3.1 guarantees that the same equation has at least $2^{|f(A)|-1}$ solutions Y over $\mathcal{P}_1(K)$.
- So $|f(A)|$ is finite, that is, $f(\mathcal{P}_{\text{fin},1}(H)) \subseteq \mathcal{P}_{\text{fin},1}(K)$.
- In fact, since K is also a cancellative monoid, we can apply the previous argument to the inverse f^{-1} of f.
- As a result, we obtain $f^{-1}[\mathcal{P}_{\text{fin},1}(K)] \subseteq \mathcal{P}_{\text{fin},1}(H)$, and therefore $\mathcal{P}_{\text{fin},1}(K)$ is contained in $f[\mathcal{P}_{\text{fin},1}(H)]$.

Outline

Main Theorem & Critical Lemma

Theorem 3.3 in [Tringali & W. 202?]

$\mathrm{Aut}(\mathcal{P}_0(H))$ is trivial for every numerical monoid H.

The proof of Theorem 3.3 relies on the following key lemma.

Lemma 3.2 in [Tringali & W. 202?]

Let H be a numerical monoid, X be a subset of H containing 0, and y be an element of H. Then $y \in X$ if and only if $H_y + X = H_y^* + X$, where

$$H_y := \{0\} \cup (H \cap \mathbb{Z}_{\geq y}) \quad \text{and} \quad H_y^* := \{0\} \cup (H_y \setminus \{y\}).$$

Remark: Both H_y and H_y^* are idempotents of $\mathcal{P}_0(H)$.

Let H be a numerical monoid and f be an automorphism of $\mathcal{P}_0(H)$. Theorem 2.3 guarantees that f restricts to an automorphism of $\mathcal{P}_{\mathrm{fin},0}(H)$.

Proof sketch of Theorem 3.3

We cite a key result from Tringali and Yan on Nathanson's Fundamental Theorem of Additive Number Theory to lay the groundwork for the proof.

Lemma 2.4 in [Tringali & Yan · Proc. AMS, 2025]

Let $\phi : \mathcal{P}_{\mathrm{fin},0}(S_1) \to \mathcal{P}_{\mathrm{fin},0}(S_2)$ be an isomorphism, where S_1 and S_2 are numerical monoids, and pick $a_1, a_2 \in S_1$. There exist $b_1, b_2 \in S_2$ s.t.

- $\phi(\{0, a_1\}) = \{0, b_1\}$ and $\phi(\{0, a_2\}) = \{0, b_2\}$.
- $\phi(\{0, a_1 + a_2\}) = \{0, b_1 + b_2\}$.

This implies there exists an automorphism $g \in \mathrm{Aut}(H)$ satisfying $f(\{0, x\}) = \{0, g(x)\}$ for all $x \in H$. Since the automorphism group of a numerical semigroup is trivial, $\{0, x\}$ is a fixed point of f for every $x \in H$. This yields our intermediate lemma:

Lemma 3.1 in [Tringali & W. 202?]

If H is a numerical monoid and f is an automorphism of $\mathcal{P}_0(H)$, then $\{0, x\}$ is a fixed point of f for every $x \in H$.

Proof sketch of Theorem 3.3

Combining Lemma 3.1 with the following proposition, we conclude that f fixes all idempotents of $\mathcal{P}_0(H)$.

Proposition 2.7 in [Tringali & W. 202?]

Let H be a monoid, and let $\mathcal{D}$ be a submonoid of $\mathcal{P}(H)$ such that

- $\{1_H, x\} \in \mathcal{D}$ for all $x \in H$, and
- every idempotent of $\mathcal{D}$ contains the identity 1_H of H.

If an automorphism f of $\mathcal{D}$ fixes each set $\{1_H, x\} \subseteq H$, then it fixes all the idempotents of $\mathcal{D}$.

For $y \in H$ and $X \in \mathcal{P}_0(H)$, it follows from Lemma 3.2 that

$$\begin{aligned} y \in X &\iff H_y + X = H_y^* + X \\ &\iff H_y + f(X) = H_y^* + f(X) \\ &\iff y \in f(X). \end{aligned}$$

This proves that $f(X) = X$, and hence f is the identity.

Outline

A new congruence relation

We introduce a binary relation that connects the automorphism group of the large power semigroup of a numerical semigroup to that of the reduced large power monoid $\mathcal{P}_0(\mathbb{N})$ of the numerical monoid $\mathbb{N}$.

Let Q be a commutative semigroup and T a subsemigroup of Q. For $a, b \in Q$, define the relation $\sim$ on Q by declaring $a \sim b$ if and only if either $a = b$ or $ax = by$ for some $x, y \in T$.

It is straightforward to verify that $\sim$ is a semigroup congruence on Q.

Consequently, we obtain the factor semigroup $Q/\sim$, and we denote by $[x]$ the congruence class of an element $x \in Q$.

In what follows, we specialize to the case where Q is the large power semigroup $\mathcal{P}(S)$ of a numerical semigroup S and T is the set of cancellative elements of $\mathcal{P}(S)$.

Note that, by a result of Tringali, the cancellative elements of $\mathcal{P}(S)$ for a numerical sgrp S are precisely the one-element subsets of S.

Main theorem

Theorem 4.3 in [Tringali & W. 202?]

$\mathrm{Aut}(\mathcal{P}(S))$ is trivial for every numerical semigroup S.

Let $f \in \mathrm{Aut}(\mathcal{P}(S))$. We establish two straightforward yet useful claims:

- The binary relation ϕ that maps the class $[X]$ of a non-empty subset X of S to $X - \min X$ is an isomorphism from $\mathcal{P}(S)/{\sim}$ to $\mathcal{P}_0(\mathbb{N})$.
- The binary relation r_f that maps the class $[X]$ of a non-empty subset X of S to the class $[f(X)]$ of $f(X)$ is an automorphism of $\mathcal{P}(S)/{\sim}$.

Using the above notation, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{P}(S)/{\sim} & \xrightarrow{\quad\phi\quad} & \mathcal{P}_0(\mathbb{N}) \\ {\scriptstyle r_f}\big\downarrow & & \big\downarrow{\scriptstyle \phi \circ r_f \circ \phi^{-1}} \\ \mathcal{P}(S)/{\sim} & \xrightarrow[\quad\phi\quad]{} & \mathcal{P}_0(\mathbb{N}) \end{array}$$

Proof sketch of Theorem 4.3

We now outline the proof that $\mathrm{Aut}(\mathcal{P}(S))$ is trivial.

- By the commutativity of our diagram, the composite function $\phi \circ r_f \circ \phi^{-1}$ is an automorphism of $\mathcal{P}_0(\mathbb{N})$.
- By Theorem 3.3, the automorphism group $\mathrm{Aut}(\mathcal{P}_0(\mathbb{N}))$ is trivial. This implies that $\phi \circ r_f \circ \phi^{-1}$ must be the identity map on $\mathcal{P}_0(\mathbb{N})$.
- Since ϕ is an isomorphism, the above fact forces r_f to be the identity automorphism on the quotient semigroup $\mathcal{P}(S)/{\sim}$. In other words, for every non-empty $X \subseteq S$, we have $[X] = [f(X)]$ in $\mathcal{P}(S)/{\sim}$.
- Combining the definition of the congruence $\sim$ with the fact that $\min X = \min f(X)$ for all $X \in \mathcal{P}(S)$, we conclude that $f(X) = X$ for every non-empty $X \subseteq S$. Thus, f is the identity automorphism, proving that $\mathrm{Aut}(\mathcal{P}(S))$ is trivial.

Outline

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